14.3 Conditional Probability and Intersections of Events

Objectives

1. Understand how to compute conditional probability.
2. Calculate the probability of the intersection of two events.
3. Use probability trees to compute conditional probabilities.
4. Understand the difference between dependent and independent events.

Rolling Two Dice:

(1,1) (2,1) (3,1) (4,1) (5,1) (6,1)
(1,2) (2,2) (3,2) (4,2) (5,2) (6,2)
(1,3) (2,3) (3,3) (4,3) (5,3) (6,3)
(1,4) (2,4) (3,4) (4,4) (5,4) (6,4)
(1,5) (2,5) (3,5) (4,5) (5,5) (6,5)
(1,6) (2,6) (3,6) (4,6) (5,6) (6,6)

What is the probability of rolling two dice and obtaining a sum that is an odd number? $\frac{1}{2}$

What is the probability of rolling a sum that is an odd number given that the sum is greater than 6?

F = sum is odd
E = sum > 6
$P(F|E) = \frac{12}{21}$

Conditional Probability

KEY POINT

Conditional probability takes into account that one event occurring may change the probability of a second event.

$P(F|E)$ ← given

DEFINITION When we compute the probability of event F assuming that the event E has already occurred, we call this the **conditional probability** of F, given E. We denote this probability as $P(F|E)$. We read $P(F|E)$ as "the probability of F given that E has occurred," or in a quicker way, "the probability of F given E."

We want the probability of the intersection outcomes, knowing that we're interested in only outcomes for set E.

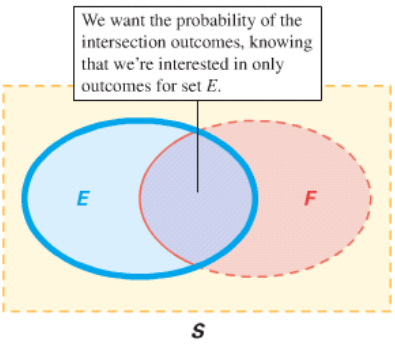


FIGURE 14.12 To compute the probability of F given E, we compare the outcomes in $E \cap F$ with the outcomes in E.

SPECIAL RULE FOR COMPUTING $P(F|E)$ BY COUNTING If E and F are events in a sample space with equally likely outcomes, then $P(F|E) = \frac{n(E \cap F)}{n(E)}$.

Examples:

1. Using a standard 52-card deck, find the probability that if you draw a card at random, it is a king given that the card is a face card.

F = drawing a king 4 ways $\frac{4}{12} = \frac{1}{3}$

E = draw a face card 12 ways

2. Two events for the sum of two dice are: E=a total greater than nine F=the total is odd

event ↓ of sample space

a. Find P(E|F)

E = 2 ways $\frac{2}{18} = \frac{1}{9}$

F = 18 ways

b. Find P(F/E) $= \frac{2}{6} = \frac{1}{3}$

In Exercises 9–12, we are drawing a single card from a standard 52-card deck. Find each probability.

9. $P(\text{heart} \mid \text{red})$ 1/2

10. $P(\text{king} \mid \text{face card})$

11. $P(\text{seven} \mid \text{nonface card})$ $\frac{4}{40} = \frac{1}{10}$

12. $P(\text{even-numbered card} \mid \text{nonface card})$ $\frac{20}{40} = \frac{1}{2}$

According to U.S. government statistics, mononucleosis (mono) is four times more common among college students than the rest of the population. Blood tests for the disease are not 100% accurate. Assume that the following table of data was obtained regarding students who came to your school's health center complaining of tiredness, a sore throat, and slight fever. Use these data to answer Exercises 23–26.

	Has Mono	Doesn't	TOTALS
Positive	72	4	76
Negative	8	56	64
TOTALS	80	60	140

a. Find the probability that a student chosen at random has mono, given that the test is positive.

$\frac{72}{76} = 0.947$

b. Find the probability that a student chosen at random does not have mono, given that the test is positive.

$\frac{4}{76} = 0.053$

c. Find the probability that the test is positive, given that the student has mono.

$\frac{72}{80} = 0.9$

The Intersection of Events

KEY POINT

We use conditional probability to find the probability of the intersection of two events.

RULE FOR COMPUTING THE PROBABILITY OF THE INTERSECTION OF EVENTS If E and F are two events, then

$$P(E \cap F) = P(E) \cdot P(F \mid E).$$

This rule says that to find the probability of $P(E \cap F)$, you first find the probability of E and then multiply it by the probability of F, *assuming that E has occurred.*

Example: Suppose that we draw two cards without replacement from a standard 52-card deck. What is the probability that both cards are red?

$\frac{1}{2} \cdot \frac{25}{51} = \frac{25}{102}$

$\frac{1}{2}$ ↑ draw a red

Probability and drawing cards. *In Exercises 27–32, assume that we draw 2 cards from a standard 52-card deck. Find the desired probabilities.*

a) First assume that the cards are drawn without replacement.

b) Next assume that the cards are drawn with replacement.

29. The probability that we draw a face card followed by a non-face card

a) $\frac{12}{52} \cdot \frac{40}{51} = \frac{40}{221}$

fc nfc

b) $\frac{12}{52} \cdot \frac{40}{52} = \frac{30}{169}$

nfc

35. We draw 3 cards without replacement from a standard 52-card deck. Find the probability of drawing three face cards.

$\frac{12}{52} \cdot \frac{11}{51} \cdot \frac{10}{50} = \frac{11}{1105}$

Playing a store's discount game. *To stimulate business, a department store has introduced a game called "Register Roulette." To play, a customer selects a first purchase and a second purchase to qualify for a discount. The customer then reaches into a container and randomly selects two game tokens. There are 50 green tokens that qualify a purchase for a 10% discount, 10 blue tokens that result in a 20% discount, and 2 red tokens that give a 50% discount.*

In Exercises 19–22, assume that you are randomly selecting two of these tokens, without replacement. Compute P(F | E).

50 G
10 B
2 R
62 tokens

19. *E*—you select a red token first.
F—the second token is green. $\frac{50}{61}$

20. *E*—you select a red token first. $\frac{1}{61}$
F—the second token is red.

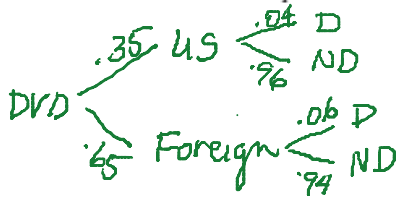
21. *E*—you select a nonred token first. $\frac{2}{61}$
F—the second token is red.

22. *E*—you select a nonred token first.
F—the second token is nonred. $\frac{59}{61}$

Probability Trees

USING TREES TO CALCULATE PROBABILITIES We can represent an experiment that happens in stages with a tree whose branches represent the outcomes of the experiment. We calculate the probability of an outcome by multiplying the probabilities found along the branch representing that outcome. We will call these trees **probability trees**.

Product reliability. *You want to purchase a DVD drive for your laptop computer. Assume that 65% of the drives are made outside the United States. Of the U.S.-made drives, 4% are defective; of the foreign-made drives, 6% are defective. Determine each probability rounded to three decimal places.*



67. The probability that the drive you purchase is U.S. made and is not defective $(.35)(.96) = 0.336$

68. The probability that the drive you purchase is foreign made and is defective $(.65)(.06) = 0.039$

Independent and Dependent Events

Definitions:

DEFINITIONS Events E and F are **independent** events if

$$P(F \mid E) = P(F).$$

If $P(F \mid E) \neq P(F)$, then E and F are **dependent**.

This definition says that if E and F are independent, then knowing that E has occurred does not influence the way we compute the probability of F.

Example: Assume we roll a pair of dice: Are the following events dependent or independent?

a. E=the total showing is greater than nine F=The total showing is even

$P(F|E) = \frac{4}{6} = \frac{2}{3}$ $P(F) = \frac{1}{2}$ dependent.

b. E=A three shows on the first die F=the total showing is even

$P(F|E)$ $\frac{3}{6} = \frac{1}{2}$ $P(F) = \frac{1}{2}$ independent.

EXAMPLE 6 *Selecting a Dormitory Room*

Brianna is taking part in a lottery for a room in one of the new dormitories at her college. She is guaranteed a space, but she will have to draw a card randomly to determine exactly which room she will have. Each card has the name of one dormitory, X, Y, or Z, and also has a two-person room number or an apartment number. Thirty percent of the available spaces are in X, 50% in Y, and 20% of the spaces are in Z.

Half of the available spaces in X are in rooms, 40% of Y's spaces are in rooms, and 30% of the spaces in Z are in rooms.

a) Draw a probability tree to describe this situation.

b) Given that Brianna selects a card for dormitory Y, what is the probability that she will be assigned to an apartment?

c) What is the probability that Brianna will be assigned an apartment in one of the three dormitories?

14.4 Expected Value

Objectives

1. Understand the meaning of expected value.
2. Calculate the expected value of lotteries and games of chance.
3. Use expected value to solve applied problems.

Expected Value

What is Expected Value? — what happens on average when you perform an experiment many times.

Sum of Two Dice: What is the expected value of the sum of two dice?

Outcomes	Probability
2	1/36
3	2/36
4	3/36
5	4/36
6	5/36
7	6/36
8	5/36
9	4/36
10	3/36
11	2/36
12	1/36

Multiply
2. $\frac{1}{36} = \frac{2}{36}$
3. $\frac{2}{36} = 6/36$
12/36
20/36
30/36
42/36
40/36
36/36

30/36
22/36
12/36
252/36 = 7

To find expected value, multiply each outcome by its probability. Then add all the answers together.

1. It costs two dollars to roll a pair of dice. If the number is odd, you win one dollar and if it is either double 1s or double 6s, you win five dollars. What is the expected value of the game? Is it fair?

charge $0.78 to make it a fair game

Outcomes	Values	Probability	mult
double 1s or 6s	+3	2/36	6/36
odd	−1	18/36	−18/36
other	−2	16/36	−32/36
			−$1.22

not fair

DEFINITIONS If a game has an expected value of 0, then the game is called **fair**. A game in which the expected value is not 0 is called an **unfair game**.

Although it would seem that you would not want to play an unfair game, in order for a casino or a state lottery to make a profit, the game has to be favored against the player.

2. It costs $1 to play a lottery where the player chooses a 3-digit number between and including 000 and 999. If the number is selected that day, then the player wins $500 (profit of $499). Find the expected value of the game. Is the game fair? What should the price of a ticket be in order for the game to be fair? $0.50 for price ticket

Let x = amt of change

price of a ticket be in order for the game to be fair? $0.50 for price ticket

Outcomes	Values		Probability	
WIN	499	$\frac{500-x}{1}$	$\frac{1}{1000}$	$\frac{500-x}{1000}$
LOSE	-1	$-x$	$\frac{999}{1000}$	$\frac{-999x}{1000}$

$$\cancel{1000} \cdot \frac{500-1000x}{\cancel{1000}} = 0 \cdot 1000$$

$$500 - 1000x = 0$$

$$\frac{500}{1000} = \frac{1000x}{1000}$$

$$\$.50 = x$$

EXAMPLE 3 *The Expected Value of a Roulette Wheel*

Although there are many ways to bet on the 38 numbers of a roulette wheel,* one simple betting scheme is to place a bet, let's say $1, on a single number. In this case, the casino pays you $35 (you also keep your $1 bet) if your number comes up and otherwise you lose the $1. What is the expected value of this bet?

Outcomes	Values	Probability	mult
W	35	$\frac{1}{38}$	$\frac{35}{38}$
L	-1	$\frac{37}{38}$	$\frac{-37}{38}$

$$\frac{-2}{38} = -0.05$$

3. Based on mortality tables, the probability of a 20-year-old male living to age 21 is 0.99. If a $1,000 one-year term life insurance policy on a 20-year-old male costs $25, what is its expected value?

Outcomes	Values	Probability

Out	Val	Prob
Lives	-25	$0.99 = -24.75$
Dies	975	$0.01 = 9.75$
		$-\$15.00$